Sec 1.6. Exercises.

1.(a). $\varphi(x^n) = \varphi(x \cdot x \cdot x \cdots x) = \varphi(x)^n$ (use induction on n)

(b). $\varphi(x^{-1}) = \varphi(x)^{-1}$: $\varphi(x^{-1})\varphi(x) = \varphi(e) = e$

11. $A \times B \cong B \times A$:

$(a,b) \longmapsto (b,a)$ gives the isomorphism

12. $(A \times B) \times C \cong A \times (B \times C)$. 如何证明?

$\varphi((a,b),c) = (a,(b,c))$

13. G. H为群. $\varphi: G \longrightarrow H$为群同态. $\varphi(G) < H$.

且 $G \cong \varphi(G)/\ker\varphi$. 因此 φ injective $\Rightarrow G \cong \varphi(G)/\ker = \varphi(G)$

14. G. H 均为群.

$\varphi: G \longrightarrow H$ 为群同态. φ单射 $\Leftrightarrow \ker\varphi$ 只含单位元.

17. G 为任意群. the map from G to itself : $g \longrightarrow g^{-1}$ 为 homo

$\Leftrightarrow \forall a\, b \quad (ab)^{-1} = a^{-1}b^{-1} \Leftrightarrow (ab)^{-1} = (ba)^{-1}$

$\Leftrightarrow ab = ba$

19. $G = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{Z}^+\}$ 理解G的结构! ~~比如. $G = \{z \in \mathbb{C} \mid z^3 =$~~

$z \longrightarrow z^k$ 为满射: 只须: $z^n = 1$ ~~$z = e^{i\frac{2\pi}{}}$~~

$z \longrightarrow z^k$ is surjective:

$\forall\ \tilde{z}.\ \tilde{z}^n = 1.$

$\forall$ $2 < k < n$ 无妨

$z_l^k = z_j^k$ ($l \neq j$ 无妨)

$z = e^{i\frac{k2\pi}{n}}$. 那么 对 $k > 1$, $\forall\ \tilde{z}$

$\tilde{z} = e^{i\frac{2k\pi}{n}}$ $e^{i\frac{2m\pi}{n}}$

$e^{i\frac{kl2\pi}{n}} = e^{i\frac{kj2\pi}{n}}$

$0 \leq k \leq n-1$

$z_1 = e^{i\frac{2m\pi}{nk}}$

往证 $\exists z_1,\ z_1^k = \tilde{z}$

$z_1^k = \tilde{z}$

$\Rightarrow e^{i\frac{k(l-j)2\pi}{n}} = 1$

~~$z^k, z_2^k, \ldots$~~ $z_0^k\ z_1^k \ldots z_{n-1}^k$ 彼此不同:

但这不是一个单射

$\frac{k(l-j)}{n} =$ 整数

$z_i^k = z_j^k \quad (z_i^k - z_j^k) = 0$

$(z_i - z_j)(z_i^{k-1} + \cdots) = 0$

例 $z = 1$. $\exists z_1, z_2$.

$n \mid k(l-j)$

$z_1^k = z_2^k = 1$

$(k = 3.\ z_1 = e^{i\frac{2\pi}{3}}$

20. Aut(G) 为 G 上的 isomorphism 全体所组成之~~群~~集合.

想证明 Aut(G) 为群.

23. G is a finite group which possess an automorphism σ. $\sigma(g)=g \Leftrightarrow g=1$. 若 $\sigma^2=Id_G$, prove that G is abelian.

证明: $\sigma^2 \cancel{(a)\sigma^2(b)=\sigma^2(ab)}$

如若 $\exists\, a,b\in G.\ ab\neq ba$ $\quad\sigma$ 为 automorphism

$$\sigma^2(ab)=ab=\sigma(\sigma(ab))$$
$$=\sigma(\sigma(a)\sigma(b))$$
$$\sigma^2(ab)$$
$$=\sigma^2(a)\sigma^2(b)$$

$$\sigma^2(ab)=\sigma^2(a)\sigma^2(b)=\sigma^2(b)\sigma^2(a)$$
$$=\sigma(\sigma(ba))$$
$$\Rightarrow \sigma(ab)=\sigma(ba)$$
$$\Rightarrow ab=ba$$

证:

$$\varphi: G\longrightarrow G$$
$$x\longmapsto x^{-1}\sigma(x).$$

$\varphi(x)=\varphi(y)$ 时. $\sigma(y)=yx^{-1}\sigma(x)$.

$$\sigma(\sigma(y))=y=\sigma(y)\sigma(x^{-1})x$$
$$\Rightarrow yx^{-1}=\sigma(yx^{-1})$$
$$\Rightarrow yx^{-1}=1_G,\ x=y$$

$\Rightarrow$. φ 为单射. 因之同时为双射.

$\exists x=y^{-1}\sigma(y)$ ..

$\forall x\ \Rightarrow \sigma(x)=\sigma(y^{-1}\sigma(y))=\sigma(y^{-1})y$

$$\sigma(y)^{-1}y=\cancel{\sigma(y)}\ (y^{-1}\sigma(y))^{-1}$$
$$=x^{-1}$$

$$\Rightarrow \sigma(x)=x^{-1}. \qquad \Rightarrow ab=ba$$

$\Rightarrow \sigma(ab)=(ab)^{-1}=b^{-1}a^{-1}=\sigma(b)\sigma(a)=\sigma(ba)$

Sect 1.7. Group actions.

3. Show the additive group $\mathbb{R}$ acts on the x,y plane $\mathbb{R}\times\mathbb{R}$ by $r.(x,y)=(x+ry,y)$.

$$(r_1+r_2)(x.y) = (x+r_1 r_2 y, y)$$

$$r_1(r_2(x,y)) = r_1(x+r_2 y, y) = (x+r_1 y + r_2 y, y)$$

$$0.(x\,y) = (x, y)$$

4. the stabilizer of a in G is a subgroup of G:

We only NTS: $\begin{cases} g_1\in Stab(a),\ g_2\in Stab(a) \Rightarrow \\ g_1 g_2^{-1}\in Stab(a). \end{cases}$

$$g_1 a = a.\ g_2 a = a \Rightarrow a = g_2^{-1}a \Rightarrow (g_1 g_2^{-1})a = g_1(g_2^{-1}(a)) = g_1 a = a$$

$\Rightarrow g_1 g_2^{-1}\in Stab(a)$. We are done.

6. $G \curvearrowright A$ faithfully $\Leftrightarrow$ the kernel of the action is the set consisting only Identity.

$G \curvearrowright A$ faithfully $\Leftrightarrow$ $\begin{matrix}\varphi: G\to S_A \\ g\mapsto \sigma_g\end{matrix}$ is injective

$$\Leftrightarrow \quad \ker\varphi = \{\sigma_g = Id\} = \{e\}$$

13. the kernal of left regular action

$$\begin{matrix}\varphi: G\longrightarrow G \\ a\longmapsto l_a(g\mapsto ag)\end{matrix}$$

$$\ker\varphi = \{h\in G \mid l_h = Id\} = e$$

因此左正则作用为一个忠实作用.

17. $G \curvearrowright G$ by left conjugation : $\sigma_g(x) = gxg^{-1}$

Deduce : $|x| = |gxg^{-1}|$ which is easy.

For any subset A of G , $|A| = |gAg^{-1}|$ (Only by construct an isomorphism).

18. $H \curvearrowright A$. $a \sim b \Leftrightarrow a = hb$. This is ~~the~~ an equivalence relation 通过这种方式得到了 Set A 的分划. 把 Set A 划成了不同的轨道.

19. $\varphi : H \longrightarrow O$ O为x所在轨道. G的子群作用, $H < G$.

$h \longmapsto hx$

φ is injective. φ is surjective. $|H| = |O|$

the orbits under the action of H partition G.

这是一个双射 ($|O| = |H|$)

$O = \{h_1x, h_2x, \cdots h_k(x)\}$

$hO = \{hh_1x, hh_2x \cdots hh_k(x)\}$

反证. $hh_ix = hh_jx \Rightarrow h_ix = h_jx$ 矛盾

则 $|O| = |H|$

$\Rightarrow |G| = n|H|$, n is the number of orbits

通过这里, 明白 ($H < G$), 则 $|H| \mid |G|$

这是由于 让 $H \curvearrowright G$下.

G就被分成了有限个轨道的无交并