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DEF): G matrix Lie gp $\subset GL(n,\mathbb{C})$

The Lie alg of G is $\mathfrak{g} = \{X \in M_n(\mathbb{C}) \mid e^{tX} \in G, \forall t \in \mathbb{R}\}$, $\mathfrak{g}$ is a real Lie algebra.

DEF) A matrix Lie gp G is complex, if its lie alg $\mathfrak{g}$ is a complex subspace of $M_n(\mathbb{C})$.

prop): If G commutative, then $\mathfrak{g}$ is commutative.

Notations: $\mathfrak{gl}(n,\mathbb{C})$ = the Lie algebra of $GL(n,\mathbb{C})$.

$\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{so}(n)$, $\mathfrak{u}(n)$, $\mathfrak{sp}(n)$, ...

prop: $\mathfrak{gl}(n,\mathbb{C}) = M_n(\mathbb{C})$

$\mathfrak{sl}(n,\mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid \operatorname{trace}(X) = 0\}$

proof: $X \in M_n(\mathbb{C})$ has tr 0, want to show $e^{tX} \in SL(n,\mathbb{C})$

$\Leftrightarrow \det(e^{tX}) = e^{\operatorname{trace}(tX)} = 1 \quad \Rightarrow X \in \mathfrak{sl}(n,\mathbb{C})$

Conversely, "$\Leftarrow$" e^{tX} has det 1. $\operatorname{trace}(X) = \frac{d}{dt} e^{t \cdot \operatorname{trace}(X)}\Big|_{t=0}$

$= \frac{d}{dt} \det(e^{tX})\Big|_{t=0} = 0$

Examples: $\mathfrak{u}(n) = \{X \in M_n(\mathbb{C}) \mid X^* = -X\}$

Lie groups and Lie algebra homomorphisms

Former: Lie group $\rightsquigarrow$ Lie algebra

Now: Lie group hom $\overset{?}{\rightsquigarrow}$ Lie algebra hom?

Thm: G, H are mat Lie gps. $\mathfrak{g}, \mathfrak{h}$ are Lie algebras. $\Phi: G \to H$ Lie gp homo.

$\exists!$ $\mathbb{R}$-linear map $\phi: \mathfrak{g} \to \mathfrak{h}$ st $\Phi(e^X) = e^{\phi(X)}$, $\forall X \in \mathfrak{g}$.

This map satisfies

① $\phi(AXA^{-1}) = \Phi(A)\Phi(X)\Phi(A)^{-1}$ $\forall X \in \mathfrak{g}$, $A \in G$

② $\phi([X,Y]) = [\phi X, \phi Y]$ (ϕ is Lie alg homo)

③ $\phi(X) = \frac{d}{dt} \Phi(e^{tX})\Big|_{t=0}$ $\forall X \in \mathfrak{g}$.

DEF) G mat Lie gp. $\mathfrak{g}$ is Lie alg of G. $A \in G$. The adjoint map $Ad_A: \mathfrak{g} \to \mathfrak{g}$ is given by $Ad_A(X) = AXA^{-1}$

prop: $G, \mathfrak{g}$ as before.

$Ad: G \longrightarrow GL(\mathfrak{g})$, $A \longmapsto Ad_A$ is a Lie group homomorphism. (continuous. $Ad(AB) = Ad_A Ad_B$.

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Furthermore, $\forall A \in G$. $Ad_A : g \to g$ is a Lie alg homomorphism. ($Ad_A[X,Y] = [Ad_A(X), Ad_A(Y)]$)

g: Lie alg. $GL(g) = \{$invertible lie maps $f: g \to g\}$

$gl(g) = \{$all linear maps $f: g \to g\}$, $[f,g] = fg - gf$

$\Rightarrow gl(g)$ is the Lie algebra of $GL(g)$.

Let $\overset{Ad}{\Phi} = Ad$, Then $\Phi: G \to GL(g)$ is a Lie gp homomorphism. $\Rightarrow \overset{ad}{\phi}: g \to gl(g)$ is the lie algebra homom

prop: $\Phi = Ad \Rightarrow \phi = ad$.

pf: from theorem 3.28. Points, $\phi(X) = \frac{d}{dt}\Phi(e^{tX})|_{t=0} = \frac{d}{dt} Ad_{e^{tX}}|_{t=0}$

$\phi(X)(Y) = \frac{d}{dt} e^{tX} Y e^{-tX}|_{t=0}$

$= e^{tX} XYe^{-tX} + e^{tX} Y e^{-tX}(-X)|_{t=0} = XY - YX$

prop: $Ad_{e^X} = e^{ad_X}$.

Ex: Let $G = GL_n(\mathbb{C})$, $g = gl(n, \mathbb{C}) = M_n(\mathbb{C})$

prop: $\forall X \in M_n(\mathbb{C})$, letting $ad_X: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $ad_X[Y] = [X,Y] \Rightarrow e^X Y e^{-X} = e^{ad_X}(Y)$

§3.6. The complexification of a real lie algebra.

DEF) V is finite-dim real v.s. The complexification of V is $V_{\mathbb{C}} = \{\underbrace{V_1 + iV_2}_{\text{formal sum}} \mid V_1, V_2 \in V\}$

Def: $i(V_1 + iV_2) = -V_2 + iV_1$

prop): g: finite-dim real lie alg. $g_{\mathbb{C}}$ is the complexification of g. (as vectorspace)

$\exists ! [\,,\,]_{g_{\mathbb{C}}}$ on $g_{\mathbb{C}}$ st:

① $g_{\mathbb{C}}$ with $[\,,\,]$ is a complex Lie algebra

② $[X,Y]_{g_{\mathbb{C}}} = [X,Y]_g$. $\forall X, Y \in g$

$g_{\mathbb{C}}$ is called the complexification of the real Lie alg g.

prop): $g \subseteq M_n(\mathbb{C})$ is a real lie alg. $\forall 0 \neq X \in g$, $iX \notin g$. $g_{\mathbb{C}} \cong \{X + iY \in M_n(\mathbb{C}) \mid X, Y \in g\}$

$gl(n, \mathbb{R})_{\mathbb{C}} \cong gl(n, \mathbb{C})$

$u(n)_{\mathbb{C}} \cong gl(n, \mathbb{C})$

pf: $u(n)_{\mathbb{C}} = \{X \in M_n(\mathbb{C}) \mid X^* = -X\}$

The exponential map

DEF) G is mat lie gp with lie algebra $\mathfrak{g}$. The exponential map for G is the map $\exp: \mathfrak{g} \to G$

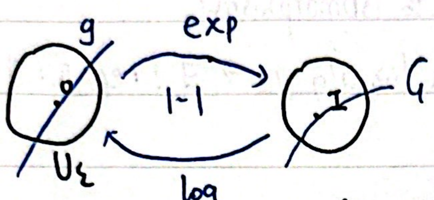
In general: $\exp: \mathfrak{g} \to G$ is neither 1-1 nor onto. However, "locally": 1-1 and onto.

Thm: For $0 < \varepsilon < \log 2$, Let $U_\varepsilon = \{X \in M_n(\mathbb{C}) \mid \|X\| < \varepsilon\}$, $V_\varepsilon = \exp(U_\varepsilon)$

$G \subseteq GL(n, \mathbb{C})$, $\mathfrak{g}$ is Lie algebra of G. Then $\exists\, 0 < \varepsilon < \log 2$. s.t $\forall A \in V_\varepsilon$, then

$A \in G \Leftrightarrow \log A \in \mathfrak{g}$.

Explanation:



$\exp: \mathfrak{g} \to G$ locally 1-1 & onto.

$\Leftrightarrow \exp(\mathfrak{g} \cap U_\varepsilon) = G \cap V_\varepsilon$

Cor: $\dim_{\mathbb{R}} \mathfrak{g} = k$, Then G is a smooth embedded submanifold $M_n(\mathbb{C})$ of dim k. Therefore

G is a Lie gp.

Cor: $G \leq GL(n, \mathbb{C})$ mat lie gp with Lie alg $\mathfrak{g}$. Then $X \in \mathfrak{g} \Leftrightarrow \exists$ smooth curve $\gamma(t)$ in G st

$\gamma(0) = I$. $\frac{d\gamma}{dt}\big|_{t=0} = X$. ($\Rightarrow$ $\mathfrak{g}$ is the tangent space of G at the identity).

Recall: $\Phi: G \to H \rightsquigarrow \phi: \mathfrak{g} \to \mathfrak{h}$

Coro: If G connected, Φ is determined by ϕ. $\Phi \xleftrightarrow{1-1} \phi$.

Basic repn theory

Recall: G ~~commutativ~~ connected. Then G commutative $\Leftrightarrow$ $\mathfrak{g}$ commute

G: mat Lie gp. $\Rightarrow G_0$ is also a matrix lie group. Lie alg of $G \cong$ Lie alg of G_0

Lie algebra is easier to learn!

Now: V is finite-dim vec sp over $\mathbb{C}$.

$GL(V)$ = group of invertible linear transformations on V ($\cong GL(n, \mathbb{C})$)

$\mathfrak{gl}(V) = \mathrm{End}(V)$ = space of all linear trans on V. (Lie alg with $[X, Y] = XY - YX$)

DEF): G matric lie gp. A repn of G is a lie group homomorphism $\Pi: G \to GL(V)$

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DEF) $\mathfrak{g}$: Lie alg. A finite-dim rep of $\mathfrak{g}$ is a lie alg homo: $\pi: \mathfrak{g} \to \mathfrak{gl}(V)$

We will always consider a rep as a linear action on a vectorspace.

Note: $\Pi: G \to GL(V)$ is group homo. $(gh).v = g.h.v$.

$\pi: \mathfrak{g} \to \mathfrak{gl}(V)$ is lie alg homo $\pi([X,Y]) = [\pi(X), \pi(Y)]$ $\Leftrightarrow [X,Y].v = X.(Y.v) - Y.(X.v)$

$= \pi(X)\pi(Y) - \pi(Y)\pi(X)$

DEF:

Π: rep of G acting on V. $W \subseteq V$ is invariant subspace if $g.W \subseteq W$. $\forall g \in G$

Typical problem: is to classify all irr repns up to isomorphisms.

prop: $\Pi: G \to GL(V)$. G is Lie gp with lie alg $\mathfrak{g}$. $\Rightarrow \exists!$ rep $\pi: \mathfrak{g} \to \mathfrak{gl}(V)$ st $\Pi(e^X) = e^{\pi(X)}$.

Moreover, $\pi(X) = \frac{d}{dt}\Pi(e^{tX})\Big|_{t=0}$.

$\pi(AXA^{-1}) = \Pi(A)\pi(X)\Pi(A)^{-1}$

prop: G: connected matrix Lie gp with lie alg $\mathfrak{g}$

① Π is irr $\Leftrightarrow \pi$ is irr

② $\Pi_1 \cong \Pi_2 \Leftrightarrow \pi_1 \cong \pi_2$

DEF: G matrix lie gp with $\mathfrak{g}$. The adjoint repn of G is $Ad: G \to GL(\mathfrak{g})$, $A \mapsto Ad_A$

The adjoint repn of $\mathfrak{g}$ is $ad: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$

$X \to ad_X$, $ad_X(Y) = [X,Y]$

Ad is rep $\Leftrightarrow Ad(AB) = Ad(A)Ad(B)$

ad is rep $\Leftrightarrow ad_{[X,Y]} = ad_X ad_Y - ad_Y ad_X$

DEF: Tensor product $f: U \to U$, $g: V \to V$ are linear map. their tensor product is: $f \otimes g: U \otimes V \to U \otimes V$

$u \otimes v \mapsto f(u) \otimes g(v)$

DEF G, H are matrix Lie gp. Π_1: rep of G on U. Π_2: rep of H on V.

The tensor product $\Pi_1 \otimes \Pi_2$ of Π_1 and Π_2 is: the rep of $G \times H$ acting on $U \otimes V$

Via $(\Pi_1 \otimes \Pi_2)(A, B) = \Pi_1(A) \otimes \Pi_2(B)$.

Note: $\Pi_1 \otimes \Pi_2: G \times H \to GL(U \otimes V)$

prop): G, H mat lie gps with lie algebras $\mathfrak{g}, \mathfrak{h}$.

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Π_1, Π_2: repns of G, H

π_1, π_2: repns of corresponding $\mathfrak{g}, \mathfrak{h}$

$\Rightarrow$ $\Pi_1 \otimes \Pi_2$: the tensor product of Π_1 and Π_2 (repn of $G \times H$).

Let $\pi_1 \otimes \pi_2$ be the corresponding Lie alg rep of $\mathfrak{g} \oplus \mathfrak{h}$. Then $(\Pi_1 \otimes \Pi_2)(X, Y) = \pi_1(X) \otimes I + I \otimes \pi_2(Y)$, $\forall X \in \mathfrak{g}, Y \in \mathfrak{h}$.

DEF: $\mathfrak{g}, \mathfrak{h}$ lie alg, π_1, π_2 are repns of $\mathfrak{g}$ and $\mathfrak{h}$ acting on U, V. The tensor product $\pi_1 \otimes \pi_2$ is the rep of $\mathfrak{g} \oplus \mathfrak{h}$ acting on $U \otimes V$ by:

$(\pi_1 \otimes \pi_2)(X, Y) = \pi_1(X) \otimes I + I \otimes \pi_2(Y)$. ie. $(X, Y)(u \otimes v) = (X.u) \otimes v + u \otimes (Y.v)$

Schur's lemma

1. V, W irr reps of a gp / lie algebra $\phi: V \to W$ is an interwining map $\Rightarrow \phi = 0$ or ϕ is an isomorphism
2. V: irr rep. $\phi: V \to V$ is an intertwing map. $\Rightarrow \phi = \lambda I$ for some $\lambda \in \mathbb{C}$
3. V, W irr reps. $\phi_1, \phi_2: V \to W$ are 2 nonzero intertwining maps. $\Rightarrow \phi_1 = \lambda \phi_2$ for some $\lambda \in \mathbb{C}$.

Note. in 2.3. the base field is $\mathbb{C}$

Repns of $sl(2, \mathbb{C})$

Goal: Find all irr repns of $sl(2, \mathbb{C})$

Standard basis: $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

lemma: π is repn of $sl(2, \mathbb{C})$. u: eigenvector of $\pi(H): V \to V$. with eigenvalue α ($\pi(H)v = \alpha v$). Then $\pi(H)\pi(X)u = (\alpha+2)\pi(X)u$. (which means $\pi(X)u$ will either be an eigenvector for $\pi(H)$ with e.v $\alpha+2$, or $\pi(X)u = 0$).

Similarly, $\pi(H)\pi(Y)u = (\alpha-2)\pi(Y)u$

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pf: $[\pi(H),\pi(X)] = \pi([H,X]) = 2\pi(X)$

$\pi(H)\pi(X) - \pi(X)\pi(H) = 2\pi(X)$

$\pi(H)\pi(X)(u) = \pi(X)\pi(H)(u) + 2\pi(X)u$

$= \pi(X)\alpha u + 2\pi(X)u = (\alpha+2)\pi(X)u$

Now. Let π be an irr rep of $sl(2,\mathbb{C})$ acting on V.

Strategy: Diagonalize $\pi(H)$, i.e. find eigenvectors spanning V.

$\mathbb{C}$ is alg closed $\Rightarrow$ $\pi(H)$ has at least one eig vector u with e.v. ν.

- By lemma $\pi(H)\pi(X)^k u = (\alpha+2k)\pi(X)^k u$ $(k\geq 0)$

  $\pi(H)$ can have at most $\dim V$ distinguish eig values. $\exists N\geq 0$ s.t

  $\pi(X)^N u \neq 0$, $\pi(X)^{N+1}u = 0$

  Let $u_0 = \pi(X)^N u$. $\lambda = \alpha + 2N$. $\Rightarrow \pi(H)u_0 = \lambda u_0$, $\pi(X)u_0 = 0$

- Let $u_k = \pi(Y)^k u_0$ for $k\geq 0$. By lemma. $\pi(H)u_k = (\lambda-2k)u_k$ $(k\geq 0)$

  $\pi(X)u_k = k(\lambda-k+1)u_{k-1}$, $(k\geq 1)$ $(*)$

  As before $u_k = 0$ for some k. Let $u_k = \pi(Y)^k u_0 \neq 0$ for $0\leq k\leq m$

  $u_{m+1} = \pi(Y)^{m+1}u_0 = 0$.

By $(*)$. $0 = \pi(X)u_{m+1} = (m+1)(\lambda-m)u_m \rightsquigarrow \lambda = m$ (integer) ≥ 0

Consider action $\pi(X)$. $\pi(Y)$ $\pi(H)$ on $u_0, \cdots u_m$

$$\left.\begin{aligned}
&\pi(H)u_k = (m-2k)u_k \\
&\pi(Y)u_k = \begin{cases} u_{k+1} & k<m \\ 0 & k=m \end{cases} \\
&\pi(X)u_k = \begin{cases} k(m-k+1)u_{k-1} & \text{if } k>0 \\ 0 & k=0 \end{cases}
\end{aligned}\right\} \maltese$$

Since $u_0, u_1, \cdots u_m$ are e.vec for $\pi(H)$ with distinct e.value. They're linear indep.

$W = \operatorname{span}\{u_0,\cdots u_m\} \subseteq V$, and W is invariant $\neq 0$. $\Rightarrow V = W$ (irreducible)

$V = \operatorname{span}\{u_0,\cdots u_m\}$, $\dim V = m+1$

$\Rightarrow$ V. V' irred rep of $sl(2,\mathbb{C})$ with same dim. then $V\cong V'$. (uniqueness)

If we define a rep by $\maltese$. then it's indeed a rep of $sl(2,\mathbb{C})$.

Thm: $\forall m\in\mathbb{Z}^+$, $\exists!$ irr representation of $sl(2,\mathbb{C})$ of dim $m+1$. (up to iso). which is given by $\maltese$.

Thm: (π, V) : finite dim representation of $sl(2,\mathbb{C})$, (not necessily irreducible).

① Every eigen value of $\pi(H)$ is integer. If v is an eigenvector of $\pi(H)$ with ev λ. $\pi(X)v=0$, then λ is a nonnegative int

②. Operators $\pi(X)$ and $\pi(Y)$ are nilpotent

③ If we define $S: V \to V$ by $S = e^{\pi(X)} e^{-\pi(Y)} e^{\pi(X)}$, then $S\pi(H)S^{-1} = -\pi(H)$

④. k is an eigen value of $\pi(H)$. then $-|k|, -|k|+2, \dots, |k|-2, |k|$ are also eigenvalues of $\pi(H)$

pf: Only NTS ③.

$$S\pi(H)S^{-1} = e^{\pi(X)} e^{-\pi(Y)} e^{\pi(X)} \pi(H) e^{-\pi(X)} e^{\pi(Y)} e^{-\pi(X)}$$

$$= Ad_{e^{\pi(X)}} Ad_{e^{-\pi(Y)}} Ad_{e^{\pi(X)}}(\pi(H)) \qquad Ad_{e^Z} = e^{ad_Z}$$

$$= e^{ad_{\pi(X)}} e^{ad_{-\pi(Y)}} e^{ad_{\pi(X)}} (\pi(H))$$

$$= e^{ad_{\pi(X)}} e^{ad_{-\pi(Y)}} (\underbrace{\pi(H) + [\pi(X), \pi(H)] + \tfrac{1}{2}[\pi(X),[\pi(X),\pi(H)]] + \cdots}_{\pi(H) - 2\pi(X)})$$

Do the same thing with $e^{ad_{-\pi(Y)}}$, $e^{ad_{\pi(X)}}$, we obtain 3.

Summary of pf of irr rep of $sl(2,\mathbb{C})$.

- u eigenvector for $sl(2,\mathbb{C})$ with eigenvalue α.

$$u \xrightarrow{\pi(X)} u' \xrightarrow{\pi(X)} u'' \xrightarrow{\pi(X)} \cdots \xrightarrow{\pi(X)} \underset{\substack{\parallel \\ u_0}}{u^{(N)}} \xrightarrow{\pi(X)} 0$$

$$0 \xleftarrow{\pi(Y)} u_m \leftarrow \cdots \xleftarrow{\pi(Y)} u_2 \xleftarrow{\pi(Y)} u_1 \xleftarrow{\pi(Y)} u_0 \leftarrow \text{Highest weight operator}$$

Next: The Baker Campbell Hausdorff formula.